Regular subrings of polynomial ring

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§1. Introduction. Throughout this article, k denotes an algebraically closed field of characteristic zero, which we fix as the ground field. Let $R:=k[u_1,\ldots,u_r]$ be a polynomial ring in r variables defined over k, and let A be a finitely generated, regular subalgebra of R. If $\dim(A)=1$, A is isomorphic to a one-parameter polynomial ring over k. However, if $\dim A \geqq 2$ there are many examples of A which are not isomorphic to a polynomial ring over k. The purpose of this article is to discuss two-dimensional, regular subalgebras contained in R. We shall recall some of necessary definitions and results.

Let V be a nonsingular projective surface defined over k and let D be a reduced effective divisor on V such that D has only normal crossings as singularities. Let $X:=V-\mathrm{Supp}(D)$. The logarithmic Kodaira dimension $\bar{\kappa}(X)$ is defined as

$$\bar{\kappa}(X)=\begin{cases}\sup\limits_{n>0}\dim\Phi_{|n(D+K_V)|}(V) & \text{if } |n(D+K_V)|\neq\phi \text{ for some } n>0\\ -\infty & \text{if otherwise.}\end{cases}$$

By definition, $\bar{\kappa}(X)=-\infty, 0, 1, 2$. We can then state the following:

THEOREM (Miyanishi-Sugie [5] and Fujita [1]). *Let V, D and X be as above. Assume that D is connected and that X contains no exceptional curves of the first kind. Then $\bar{\kappa}(X)=-\infty$ if and only if X contains a cylinderlike open set $U\cong U_0\times\mathbf{A}^1_k$, where U_0 is a curve.*

THEOREM (Miyanishi [2]). *Let $X = \mathrm{Spec}(A)$ be a nonsingular affine surface defined over k. Then X is isomorphic to the affine plane $\mathbb{A}^2_k$ if and only if $A^* = k^*$, A is a unique factorization domain, and $\bar{\kappa}(X) = -\infty$.*

Let X be a nonsingular affine surface and let C be a nonsingular curve. We say that X has an $\mathbb{A}^1$-*fibration over* C if there exists a surjective morphism $f : X \longrightarrow C$ such that general fibers of f are isomorphic to the affine line $\mathbb{A}^1_k$. Then the following conditions are equivalent to each other:

(i) $\bar{\kappa}(X) = -\infty$,

(ii) X contains a cylinderlike open set,

(iii) X has an $\mathbb{A}^1$-fibration over a curve C.

§ 2. Affine surfaces with $\mathbb{A}^1$-fibrations

2.1. Let $X = \mathrm{Spec}(A)$ be a nonsingular affine surface. Then A is contained in a polynomial ring (defined over k) if and only if there exists a dominant morphism $\rho : \mathbb{A}^2_k \longrightarrow X$. Assume that A is contained in a polynomial ring. Then we know that:

(1) A^* (= the set of invertible elements of A) $= k^*$,

(2) there is an $\mathbb{A}^1$-fibration $f : X \longrightarrow C$, where $C \cong \mathbb{A}^1_k$ or $C \cong \mathbb{P}^1_k$.

Let $f : X \longrightarrow C$ be an $\mathbb{A}^1$-fibration such that $C \cong \mathbb{A}^1_k$ or $C \cong \mathbb{P}^1_k$. Let F be a fiber. If F is irreducible and reduced, then $F \cong \mathbb{A}^1_k$. Otherwise, F_{red} is a disjoint union of irreducible components, each of which is isomorphic to $\mathbb{A}^1_k$. For every point P of C, let μ_P be the number of irreducible components of the fiber $f^*(P)$. Then we have:

$$\operatorname{rank}_{\mathbb{Q}} \operatorname{Pic}(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = \begin{cases} 1 + \sum\limits_{P\in C} (\mu_P - 1) & \text{if } C \cong \mathbf{P}^1_k \\ \sum\limits_{P\in C} (\mu_P - 1) & \text{if } C \cong \mathbb{A}^1_k \,. \end{cases}$$

2.2. THEOREM. Let A be a regular subalgebra of $R := k[u_1, u_2]$ such that R is a finite A-module. Then A is isomorphic to a polynomial ring in two variables over k.

Proof. There are an $\mathbb{A}^1$-fibration $f : X \longrightarrow C$ and a dominant morphism $\rho : \mathbb{A}^2_k \longrightarrow X$ induced by the inclusion $A \hookrightarrow R$. Then $nD \sim 0$ for every divisor D on X, where $n := \deg(\rho)$. Hence $C \cong \mathbb{A}^1_k$, and if $f^*(P)$ is singular, i.e., $f^*(P)$ is reducible or non-reduced, then $f^*(P) = n_P C_P$, where $C_P \cong \mathbb{A}^1_k$, $n_P \geq 2$ and $n_P | n$. Suppose f has a singular fiber $f^*(P) = n_P C_P$. Choose an inhomogeneous coordinate t of C so that P is defined by $t = 0$. Then $t = \tau^{n_P}$ for $\tau \in R$. Let $A' = A \underset{k[t]}{\otimes} k[\tau] \cong A[\tau] \subset R$, let $\tilde{A}$ be the normalization of A' and let $\tilde{X} := \operatorname{Spec}(\tilde{A})$. Then $\rho : \mathbb{A}^2_k \longrightarrow X$ factors as

$$\rho : \mathbb{A}^2_k \xrightarrow[\rho_1]{} \tilde{X} \xrightarrow[\rho_2]{} X,$$

where $\tilde{X}$ is a nonsingular affine surface with an $\mathbb{A}^1$-fibration $\tilde{f} : \tilde{X} \longrightarrow \tilde{C}$, and $\tilde{f}$ has a singular fiber with n_P irreducible components. This is a contradiction. Hence $f : X \longrightarrow C$ is an $\mathbb{A}^1$-bundle over $\mathbb{A}^1_k$. Thus, $X \cong \mathbb{A}^2_k$. Q.E.D.

2.3. Let $f : X \longrightarrow C$ be an $\mathbb{A}^1$-fibration over a curve C. Let $f^*(P) = \sum_{i=1}^{s} n_i C_i$ be a singular fiber. $f^*(P)$ is called a singular *fiber of the first kind* if $s \geq 2$ and $n_i = 1$ for some i; $f^*(P)$ is called *a singular fiber of the second kind* if $n_i \geq 2$ for every i. The integer $\mu := \text{G.C.D.}(n_1, \ldots, n_s)$ is called *the multiplicity* of $f^*(P)$. If $\mu > 1$, $f^*(P)$ is called *a multiple fiber*.

THEOREM. *Let* $X = \operatorname{Spec}(A)$ *be a nonsingular affine surface with an* $\mathbf{A}^1$*-fibration* $f : X \longrightarrow C$, *where* $C \cong \mathbf{A}^1_k$. *Then* A *is contained in a polynomial ring if and only if* f *has at most one singular fiber of the second kind.*

For the proof, we use the following:

LEMMA. *Consider a Diophantine equation*

$$(*) \qquad x_1^{a_1} \cdots x_m^{a_m} - y_1^{b_1} \cdots y_n^{b_n} = 1 ,$$

where a_i*'s and* b_j*'s are integers* ≥ 2. *Then a non-constant solution of* $(*)$ *in* $R = k[u_1, \ldots, u_r]$ *is one of the following:*

(1) $x_i = 0$ *for some* i *and* $y_j = c_j \in k$ *for every* j, *where* $c_1^{b_1} \cdots c_n^{b_n} = -1$;

(2) $y_j = 0$ *for some* j *and* $x_i = c_i \in k$ *for every* i, *where* $c_1^{a_1} \cdots c_m^{a_m} = 1$.

Proof of Theorem. The "only if" part. Suppose $f^*(P)$ and $f^*(Q)$ are singular fibers of the second kind. Let $\rho : \mathbf{A}^2_k \longrightarrow X$ be a dominant morphism. Then $\rho^*f^*(P)$ and $\rho^*f^*(Q)$ are defined by

$$f_1^{a_1} \cdots f_m^{a_m} = 0 \quad \text{and} \quad g_1^{b_1} \cdots g_n^{b_n} = 0,$$

respectively, where $a_i, b_j \geq 2$ and $f_i, g_j \in k[u_1, u_2] - k$ for every i and j. Choose a coordinate t of C so that P, Q are defined by $t = 0, 1$, respectively. Then we have

$$f_1^{a_1} \cdots f_m^{a_m} - g_1^{b_1} \cdots g_n^{b_n} = 1 .$$

This is a contradiction. The "if" part. Replacing X by an affine open subset, we may assume that f has no singular fibers of the first kind and that the unique singular fiber of the second kind (if any) is of the form $f^*(P) = n_P C_P$, where $n_P \geqq 2$. Let $C = \mathrm{Spec}(k[t])$ and let $\tilde{C} = \mathrm{Spec}(k[\tau])$, where P is given by $t = 0$ and $t = \tau^{n_P}$. Let $\tilde{X}$ be the normalization of $X \times_C \tilde{C}$. Then $\tilde{X}$ is nonsingular, and the projection $f : X \longrightarrow X \times_C \tilde{C} \longrightarrow \tilde{C}$ is an $\mathbf{A}^1$-fibration such that $\tilde{f}^*(\tilde{P})$ is the unique singular fiber, where $\tilde{P}$ lies over P and $\tilde{f}^*(\tilde{P})$ is of the first kind. Then $\tilde{X}$ contains an open set which is an $\mathbf{A}^1$-bundle over $\tilde{C} \cong \mathbf{A}_k^1$. Thus we obtain a dominant morphism $\rho : \mathbf{A}_k^2 \longrightarrow X$.

Q.E.D.

Under the situation of Theorem, $\mathrm{Pic}(X)_{tor}$ is a cyclic group.

2.4. THEOREM. Let $X = \mathrm{Spec}(A)$ be a nonsingular affine surface with an $\mathbf{A}^1$-fibration $f : X \longrightarrow C$, where $C \cong \mathbf{P}_k^1$. Then we have:

(1) Assume that A is contained in a polynomial ring. Then f *has at most three multiple fibers*. *If* f *has three multiple* fibers their multiplicities $\{\mu_1, \mu_2, \mu_3\}$ are, up to permutations, $\{2,2,n\}$ $(n \geq 2)$, $\{2,3,3\}$, $\{2,3,4\}$ and $\{2,3,5\}$.

(2) Assume that f satisfies the conditions:

(i) f has no singular fibers of the second kind but at most three multiple fibers with single irreducible components;

(ii) *if* f *has three multiple fibers, their multiplicities* $\{\mu_1, \mu_2, \mu_3\}$ *are, up to permutations,* $\{2,2,n\}$ $(n \geq 2)$, $\{2,3,3\}$, $\{2,3,4\}$ *and* $\{2,3,5\}$.

Then A *is contained in a polynomial ring.*

For the proof, we need the following:

LEMMA. (1) *Let* $S := S_{p_1,p_2,p_3}$ *be a hypersurface in* $\mathbb{A}_k^3 := \operatorname{Spec}(k[x_1,x_2,x_3])$ *defined by*

$$x_1^{p_1} + x_2^{p_2} + x_3^{p_3} = 0 ,$$

and let $S^* := S - (0,0,0)$, *where* p_1, p_2 *and* p_3 *are integers* ≥ 2. *If* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \leq 1$ *then there are no non-constant morphisms from* $\mathbb{A}_k^r$ *to* S^*. *If* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1$ *then there is a dominant morphism from* $\mathbb{A}_k^2$ *to* S^*.

(2) *Let* $\Sigma := \Sigma_{p_1,p_2,p_3,p_4}$ *be a subvariety in* $\mathbb{A}_k^4 := \operatorname{Spec}(k[x_1, x_2,x_3,x_4])$ *of codimension* 2 *defined by*

$$x_1^{p_1} + x_2^{p_2} + x_3^{p_3} = ax_1^{p_1} + x_2^{p_2} + x_4^{p_4} = 0 ,$$

and let $\Sigma^* := \Sigma - (0)$, *where* p_i $(1 \leq i \leq 4)$ *is an integer* ≥ 2 *and* $a \in k - \{0,1\}$. *If* $\{p_1,p_2,p_3,p_4\}$ *is one of the following quadruplets:* $\{2,2,2,n\}$ $(n \geq 2)$, $\{2,2,3,3\}$, $\{2,2,3,4\}$ *and* $\{2,2,3,5\}$, *then there are no non-constant morphisms from* $\mathbb{A}_k^r$ *to* Σ^*.

Proof of Theorem. (1) Suppose f has three or more multiple fibers, and let $f^*(P_i)$ $(1 \leq i \leq 3)$ be multiple fibers with respective multiplicities $\mu_i \geqq 2$. Let $\rho : \mathbb{A}_k^2 \longrightarrow X$ be a dominant morphism. Then $f\cdot\rho(\mathbb{A}_k^2)$ is isomorphic to $\mathbb{A}_k^1$ or $\mathbb{P}_k^1$. If $f\cdot\rho(\mathbb{A}_k^2) \cong \mathbb{A}_k^1$, then two of P_i's are in $f\cdot\rho(\mathbb{A}_k^2)$. This leads to a contradiction by Theorem 2.3. Hence $f\cdot\rho(\mathbb{A}_k^2) = C$. Write $f^*(P_i) = \mu_i F_i$. Then $\rho^* F_i$ is defined by $f_i = 0$ in $\mathbb{A}_k^2$, where $f_i \in k[u_1,u_2]$. Then we have

$$\frac{f_3^{\mu_3}}{f_1^{\mu_1}} = a\,\frac{f_2^{\mu_2}}{f_1^{\mu_1}} + b \quad \text{with } a, b \in k^* .$$

Since $\rho^*(F_i) \cap \rho^*(F_j) = \phi$ if $i \neq j$, we have a non-constant morphism

$$\varphi : \mathbb{A}_k^2 \longrightarrow S^*_{\mu_1,\mu_2,\mu_3} .$$

Hence $\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} > 1$. Then $\{\mu_1,\mu_2,\mu_3\}$ is, up to permutations, one of the triplets: $\{2,2,n\}$ $(n \geqq 2)$, $\{2,3,3\}$, $\{2,3,4\}$ and $\{2,3,5\}$. Suppose f has four multiple fibers $f^*(P_i) = \mu_i F_i$ $(1 \leqq i \leqq 4)$ with $\mu_i \geqq 2$. Then we obtain relations of the form

$$f_1^{\mu_1} + f_2^{\mu_2} + f_3^{\mu_3} = af_1^{\mu_1} + f_2^{\mu_2} + f_4^{\mu_4} = 0, \quad a \in k-\{o,1\} ,$$

where $f_i \in k[u_1,u_2]-\{0\}$. Then we have a non-constant morphism

$$\psi : \mathbb{A}_k^2 \longrightarrow \Sigma^*_{\mu_1,\mu_2,\mu_3,\mu_4} ,$$

which is a contradiction.

(2) Replacing X by an affine open subset, we may assume that f has no singular fibers of the first kind. Suppose f has at most two multiple fibers, say, for example, two multiple fibers $f^*(P_i)$ $(i = 1, 2)$. Let $X' := X - f^{-1}(P_2)$ and $C' := C - \{P_2\}$.

Then $f' = f|_{X'} : X' \longrightarrow C'$ is an $\mathbb{A}^1$-fibration over $C' \cong \mathbb{A}^1_k$ with one singular fiber $f^*(P_1)$ of the second kind. Then we are done by Theorem 2.3. Suppose f has three multiple fibers $f^*(P_i) = \mu_i F_i$ with $\{\mu_1, \mu_2, \mu_3\}$ as specified in the statement. Consider the case where $\{\mu_1, \mu_2, \mu_3\} = \{2,2,n\}$. Let $\tau : C' \longrightarrow C$ be a double covering ramified over P_1 and P_2, let X' be the normalization of $X \times_C C'$, and let $f' : X' \longrightarrow C'$ be the natural $\mathbb{A}^1$-fibration over $C' \cong \mathbb{P}^1_k$. Then f' has only two multiple fibers $f'^*(Q_i)$ $(i = 1, 2)$ of multiplicity n, where $\tau^{-1}(P_3) = \{Q_1, Q_2\}$. Then we are done by the former case. The cases where $\{\mu_1, \mu_2, \mu_3\} = \{2,3,3\}$ or $\{2,3,4\}$ are dealt with by a similar fashion;

$$\{\mu_1, \mu_2, \mu_3\} = \{2,3,3\} \xrightarrow[\text{covering}]{\text{triple}} \{2,2,2\} \longrightarrow \text{the former case,}$$

$$\{\mu_1, \mu_2, \mu_3\} = \{2,3,4\} \xrightarrow[\text{covering}]{\text{double}} \{2,3,3\} \longrightarrow \text{the former case.}$$

In the case where $\{\mu_1, \mu_2, \mu_3\} = \{2,3,5\}$, we know by the theory of Kleinian singularities that there exists a ramified covering $\tau : C' \longrightarrow C$ of degree 60 with 30 points over P_1 with ramification index 2, 20 points over P_2 with ramification index 3 and 12 points over P_3 with ramification index 5, where $C' \cong \mathbb{P}^1_k$. Let X' be the normalization of $X \times_C C'$ and $f' : X' \longrightarrow C'$ be the natural $\mathbb{A}^1$-fibration. Then f' has no multiple fibers of the second kind. So, we are done. Q.E.D.

§ 3. <u>Surfaces with $\mathbb{A}^1_*$-fibrations</u>

3.1. We denote by $\mathbb{A}^1_*$ the affine line $\mathbb{A}^1_k$ with one point deleted off. Let X be a nonsingular surface and let C be a nonsingular curve. An $\mathbb{A}^1_*$-fibration on X over C is a surjective morphism $f : X \longrightarrow C$ such that general fibers of f are isomorphic to $\mathbb{A}^1_*$ and every singular fiber (if any) is of the form $f^*(P) = n_P C_P$, where $n_P \geqq 2$ and $C_P \cong \mathbb{A}^1_*$. The morphism f (or X itself) is called also an $\mathbb{A}^1_*$-fiber space. A normal compactification of X is a nonsingular projective surface V containing X as a dense open set such that $V-X$ consists of nonsingular irreducible curves crossing normally each other.

An $\mathbb{A}^1_*$-fiber space $f : X \longrightarrow C$ has the following normal compactification $\varphi : V \longrightarrow Y$ such that:

(i) X and C are dense open subsets of V and Y, respectively;

(ii) φ is a $\mathbb{P}^1$-fibration and $\varphi|_X = f$;

(iii) $V-X$ contains no exceptional curves of the first kind contained in fibers of φ;

(iv) there are two cross-sections M_0, M_∞ such that M_0, $M_\infty \subset V-X$, $M_0 \cap M_\infty = \phi$ and the other components of $V-X$ are contained in fibers of φ.

3.2. LEMMA. Let X be a nonsingular, quasi-projective surface with an effective, separated G_m-action. Assume that X has no fixed points. Let $f : X \longrightarrow C := X/G_m$ be the quotient morphism. Then we have:

(1) C is a nonsingular curve, and X is an $\mathbb{A}^1_*$-fiber space over C;

(2) $f^*(P)$ <u>is a multiple fiber with multiplicity</u> μ_P <u>if and only if the stabilizer group</u> σ_x <u>is a cyclic group of order</u> μ_P <u>for a point</u> x <u>of</u> $f^{-1}(P)$.

3.3. Let $S = S_{p_1,p_2,p_3}$ be as in Lemma 2.4. Let d = L.C.M. (p_1,p_2,p_3) and define integers q_i by $d = p_iq_i$. Then G_m acts effectively on S^* by $t(x_1,x_2,x_3) = (t^{q_1}x_1, t^{q_2}x_2, t^{q_3}x_3)$. Let $f : S^* \longrightarrow C$ be the quotient morphism, where C is a complete curve. Define integers p_i' $(1 \leq i \leq 3)$ by

$$p_1' = \frac{p_1}{(q_2,q_3)}, \quad p_2' = \frac{p_2}{(q_1,q_3)} \quad \text{and} \quad p_3' = \frac{p_3}{(q_1,q_2)}.$$

Then we have:

LEMMA. (1) <u>The genus</u> $g(C)$ <u>of</u> C <u>is given as</u>

$$g(C) = \frac{d^2}{2q_1q_2q_3} - \frac{d}{2}\left(\frac{(q_1,q_2)}{q_1q_2} + \frac{(q_2,q_3)}{q_2q_3} + \frac{(q_3,q_1)}{q_3q_1}\right) + 1.$$

(2) f <u>has no multiple fibers but possibly</u> $\frac{d(q_1,q_2)}{q_1q_2}$ <u>fibers with multiplicity</u> (q_1,q_2), $\frac{d(q_2,q_3)}{q_2q_3}$ <u>fibers with multiplicity</u> (q_2,q_3) <u>and</u> $\frac{d(q_3,q_1)}{q_3q_1}$ <u>fibers with multiplicity</u> (q_3,q_1).

(3)
$$g(C)\begin{cases} = 0 \\ = 1 \\ > 1 \end{cases} \iff \frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{p_3'} \begin{cases} > 1 \\ = 1 \\ < 1. \end{cases}$$

(4)
$$\bar{\kappa}(S^*) = \begin{cases} -\infty \\ 0 \\ 1 \end{cases} \iff \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \begin{cases} > 1 \\ = 1 \\ < 1. \end{cases}$$

(5) <u>Assume that</u> $k = \mathbb{C}$. <u>Let</u> U <u>be the universal covering space of</u> S^*. <u>Then we have</u>:

$$U \cong \begin{cases} \mathbb{C}^2-(0) \\ \mathbb{C}^2 \\ \mathbb{C} \times D \end{cases} \Longleftrightarrow \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} \begin{cases} > 1 \\ = 1 \\ < 1 \end{cases},$$

<u>where</u> D <u>is a unit disc</u>.

(6) <u>Suppose</u> $p_1 \leqq p_2 \leqq p_3$. <u>Then we have</u>:

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} > 1 \Longleftrightarrow \{p_1,p_2,p_3\} = \{2,2,n\}\ (n \geqq 2),\ \{2,3,3\},\ \{2,3,4\}\ \underline{\text{or}}\ \{2,3,5\}.$$

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} = 1 \Longleftrightarrow \{p_1,p_2,p_3\} = \{2,3,6\},\ \{2,4,4\}\ \underline{\text{or}}\ \{3,3,3\}.$$

<u>Proof</u>. We prove only the assertion (4). Let $\varphi : V \longrightarrow C$ be the normal compactification of $f : S^* \longrightarrow C$ as described in 3.1, where $X = S^*$ and $Y = C$. Let $\Phi_1, \dots, \Phi_N$ exhaust all multiple fibers of φ. The following description of V is found in Orlik-Wagreich [6]. Let Φ be a multiple fiber of multiplicity α, say $\alpha = (q_1,q_2) > 1$. Define an integer β by the conditions: $0 < \beta < \alpha$ and $q_3\beta \equiv 1 \pmod{\alpha}$. Define positive integers $b_1, \dots, b_s$ $(\geqq 2)$ by a continued fraction

$$\frac{\alpha}{\alpha-\beta} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_s}}}.$$

Define integers α_i, β_i for each fiber Φ_i, and let

$$b = \frac{d}{q_1 q_2 q_3} - \sum_{i=1}^{N} \frac{\beta_i}{\alpha_i} .$$

Then $(M_0^2) = -b-N$, $(M_\infty^2) = b$ and the dual graph of Φ is

$$\underset{M_0}{\circ} \text{---} \underset{-b_1}{\circ} \text{---} \underset{-b_2}{\circ} \text{---} \cdots\cdots \text{---} \underset{-b_s}{\circ} \text{---} \underset{\bar{F}}{\circ} \text{---} \circ \text{---} \cdots\cdots \text{---} \circ \text{---} \underset{M_\infty}{\circ} ,$$

where each irreducible component of Φ is a nonsingular rational curve, $\Phi \cap S^* = \alpha F$, $\bar{F}$ is the closure of F in V, and $\bar{F}$ is the unique exceptional curve of the first kind in Φ.

Let D be the reduced effective divisor on V such that $\mathrm{Supp}(D) = V - S^*$. If $g(C) = 1$, φ has no multiple fibers. Hence $D+K_V \sim 0$, whence $\bar{\kappa}(S^*) = 0$. In general, we have

$$D+K_V \sim \sum_{i=1}^{N} \Phi_i - \sum_{i=1}^{N} \bar{F}_i + \varphi^*(K_C) \geqq \sum_{i=1}^{N} (1-\frac{1}{\alpha_i})\Phi_i + \varphi^*(K_C) .$$

Let $A := (\sum_{i=1}^{N} (1-\frac{1}{\alpha_i})\Phi_i + \varphi^*(K_C)\cdot M_0)$. Then we have

$$A = (\sum_{P\in C} (1-\frac{1}{\alpha_P})\varphi^*(P) + \varphi^*(K_C)\cdot M_0)$$

$$= \frac{d^2}{q_1 q_2 q_3}(1-\frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3}) .$$

Hence we obtain our conclusion. Q.E.D.

3.4. Let $\Sigma = \Sigma_{p_1,p_2,p_3,p_4}$ be as in Lemma 3.4. Then Σ^* has an effective separated action of G_m defined by

$$t(x_1,x_2,x_3,x_4) = (t^{q_1}x_1, t^{q_2}x_2, t^{q_3}x_3, t^{q_4}x_4) .$$

where $d = \mathrm{L.C.M.}(p_1,p_2,p_3,p_4)$ and $d = p_i q_i$. Let $f : \Sigma^* \longrightarrow C$ be the quotient morphism. Then we have:

LEMMA. (1) C *is a complete nonsingular curve of genus*

$$g(C) = \frac{d^3}{q_1q_2q_3q_4} - \frac{d^2}{2}\left\{\frac{(q_1,q_2,q_3)}{q_1q_2q_3}+\frac{(q_1,q_2,q_4)}{q_1q_2q_4}+\frac{(q_1,q_3,q_4)}{q_1q_3q_4}+\frac{(q_2,q_3,q_4)}{q_2q_3q_4}\right\} + 1 .$$

(2) f *has no multiple fibers but possibly* $\frac{d^2(q_1,q_2,q_3)}{q_1q_2q_3}$ *fibers with multiplicity* (q_1,q_2,q_3), $\frac{d^2(q_1,q_2,q_4)}{q_1q_2q_4}$ *fibers with multiplicity* (q_1,q_2,q_4), $\frac{d^2(q_1,q_3,q_4)}{q_1q_3q_4}$ *fibers with multiplicity* (q_1,q_3,q_4) *and* $\frac{d^2(q_2,q_3,q_4)}{q_2q_3q_4}$ *fibers with multiplicity* (q_2,q_3,q_4).

(3) *We have the following table*:

$\{p_1,p_2,p_3,p_4\}$	$g(C)$	multiple fibers of f
$\{2,2,2,2s\}$	1	4 fibers with multiplicity s
$\{2,2,2,2s+1\}$	0	4 fibers with multiplicity $2s+1$
$\{2,2,3,3\}$	2	no multiple fibers
$\{2,2,3,4\}$	0	2 fibers with multiplicity 2 4 fibers with multiplicity 3
$\{2,2,3,5\}$	0	2 fibers with multiplicity 5 2 fibers with multiplicity 3

3.5. *Proof of Lemma* 2.4. (1) Let $\varphi : \mathbb{A}^r_k \longrightarrow S^*$ be a non-constant morphism (if it exists at all). Then $\varphi(\mathbb{A}^r_k)$ is not contained in a fiber of f. Thus $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \neq 1$ because $g(C) = 1$. Suppose $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$. If φ is dominant, we may assume $r = 2$. Then $-\infty = \bar{\kappa}(\mathbb{A}^2_k) \geqq \bar{\kappa}(S^*) = 1$, which is a contradiction. Hence $\varphi(\mathbb{A}^r_k)$ is a rational curve with at most one place at infinity. Let $\psi = f\cdot\varphi$. Then $\psi(\mathbb{A}^r_k) \cong \mathbb{A}^1_k$ or $\mathbb{P}^1_k$, and $C \cong \mathbb{P}^1_k$. Then we can show that f has three or more multiple fibers. If $\psi(\mathbb{A}^r_k) \cong \mathbb{A}^1_k$, we obtain a contradiction by making use of Lemma 2.3. If $\psi(\mathbb{A}^r_k) \cong \mathbb{P}^1_k$, the Riemann-Hurwitz formula implies

$$\sum_{i=1}^{N} (1 - \frac{1}{\alpha_i}) \leqq 2,$$

where N and α_i's are as in the proof of Lemma 3.3. Hence $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geqq 1$, a contradiction.

(1') Suppose $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1$. Let $\{p_1,p_2,p_3\} = \{2,2,2\}$. Then we can easily find a solution $\{x_i = f_i;\ 1 \leqq i \leqq 3\}$ of $x_1^2+x_2^2+x_3^2 = 0$ in $R = k[u_1,u_2]$ such that $\mathrm{tr.deg}_k k(f_1,f_2,f_3) = 2$. Then there is a dominant morphism $\varphi : \mathbb{A}^2_k \longrightarrow S^*_{2,2,2}$. Since $\mathbb{A}^2_k$ is simply connected and there is a finite étale morphism π : $\mathbb{A}^2_k-(0) \longrightarrow S^*_{2,2,2}$, φ factors as $\varphi = \pi\cdot\tilde{\varphi}$, where $\tilde{\varphi} : \mathbb{A}^2_k \longrightarrow \mathbb{A}^2_k-(0)$ is a morphism. In other cases, there is a finite etale morphism $\pi^* : \mathbb{A}^2_k-(0) \longrightarrow S^*_{p_1,p_2,p_3}$. Then $\varphi^* = \pi^*\cdot\tilde{\varphi} : \mathbb{A}^2_k \longrightarrow S^*_{p_1,p_2,p_3}$ is a dominant morphism.

(2) The proof depends on Lemma 3.4, (3). Q.E.D.

3.6. In the rest of this section, we retain the assumptions and the notations of 3.1. We assume that $Y = C \cong \mathbb{P}^1_k$ and that the dual graph of a fiber $\varphi^*(P)$ is a linear chain for every point P of C, where every irreducible component of $\varphi^*(P)$ is a nonsingular rational curve. We assume that $(M_0^2) < 0$. Indeed, $(M_0^2) < 0$ or $(M_\infty^2) < 0$ provided f has multiple fibers. Let

$f^*(P_i) = \alpha_i C_i$ $(1 \leq i \leq N)$ exhaust all multiple fibers of f, where $C_i \cong \mathbf{A}^1_*$, $\alpha_i \geq 2$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N$.

LEMMA. (1) $\bar{\kappa}(X) = -\infty$ <u>if and only if either</u> $N \leq 2$ <u>or</u> $N = 3$ <u>and</u> $\{\alpha_1,\alpha_2,\alpha_3\}$ <u>is one of the following triplets</u>: $\{2,2,n\}$ $(n \geq 2)$, $\{2,3,3\}$, $\{2,3,4\}$, $\{2,3,5\}$.

(2) $\bar{\kappa}(X) = 0$ <u>if and only if either</u> $N = 4$ <u>and</u> $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\} = \{2,2,2,2\}$ <u>or</u> $N = 3$ <u>and</u> $\{\alpha_1,\alpha_2,\alpha_3\}$ <u>is one of the triplets</u>: $\{2,3,6\}$, $\{2,4,4\}$, $\{3,3,3\}$, <u>The logarithmic pluri-genera are given as follows</u>:

$\bar{P}_1(X) = 0$, $\bar{P}_2(X) = 1$ <u>if</u> $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\} = \{2,2,2,2\}$;

$\bar{P}_i(X) = 0$ $(1 \leq i \leq 5)$, $\bar{P}_6(X) = 1$ <u>if</u> $\{\alpha_1,\alpha_2,\alpha_3\} = \{2,3,6\}$;

$\bar{P}_i(X) = 0$ $(1 \leq i \leq 3)$, $\bar{P}_4(X) = 1$ <u>if</u> $\{\alpha_1,\alpha_2,\alpha_3\} = \{2,4,4\}$;

$\bar{P}_i(X) = 0$ $(i = 1, 2)$, $\bar{P}_3(X) = 1$ <u>if</u> $\{\alpha_1,\alpha_2,\alpha_3\} = \{3,3,3\}$.

(3) $\bar{\kappa}(X) = 1$ <u>if and only if</u> $N \geq 3$ <u>and</u>

$$\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_N} < N-2.$$

3.7. Let $\tau : V \longrightarrow \bar{V}$ be the contraction of V to a relatively minimal surface $\bar{V}$ such that $(\bar{M}_0^2) = (M_0^2) < 0$ and $(\bar{M}_\infty^2) = (M_\infty^2) + N$, wnere $\bar{M}_0 = \tau(M_0)$ and $\bar{M}_\infty = \tau(M_\infty)$. Let $\rho : V \longrightarrow \tilde{V}$ be the contraction of V to a relatively minimal ruled surface $\tilde{V}$ such that $(\tilde{M}_0^2) = (M_0^2)+N$ and $(\tilde{M}_\infty^2) = (M_\infty^2)$, where $\tilde{M}_0 = \rho(M_0)$ and $\tilde{M}_\infty = \rho(M_\infty)$. Then τ and ρ are uniquely determined.

THEOREM. <u>Assume that</u> $N = 3$, $m := (M_\infty^2) \geq 0$ <u>and</u> $\{\alpha_1,\alpha_2,\alpha_3\}$ <u>is one of the triplets</u>: $\{2,2,n\}$ $(n \geq 2)$, $\{2,3,3\}$, $\{2,3,4\}$, $\{2,3,5\}$. <u>Then</u> $\bar{\kappa}(X) = -\infty$, <u>but</u> X <u>contains no cylinderlike open sets</u>.

3.8. There are examples of $\mathbf{A}^1_*$-fiber spaces over $\mathbf{P}^1_k$ with $m < 0$. For example, $X = S^*_{p_1,p_2,p_3}$, where $\{p_1,p_2,p_3\}$ is one of the triplets: $\{2,2,n\}$ $(n \geqq 2)$, $\{2,3,3\}$, $\{2,3,4\}$, $\{2,3,5\}$, for which $m = -1$.

THEOREM. (1) $S^*_{p_1,p_2,p_3}$ <u>contains no cylinderlike open sets</u> <u>if</u> $\{p_1,p_2,p_3\} \neq \{2,2,n\}$ $(n \geqq 2)$.

(2) $S^*_{2,2,n}$ $(n \geqq 2)$ <u>contains a cylinderlike open set</u>.

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